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Constraints on a scale invariant power spectrum from superinflation in LQC

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The computation of the spectrum of primordial perturbations, generated by a scalar field during the superinflationary phase of Loop Quantum Cosmology, is revisited. The calculation is performed for two different cases. The first considers the dynamics of a massless field and it is found that scale invariance can only be achieved under a severe fine tuning. The second assumes that the field evolves with a constant ratio between kinetic and potential energy, i.e. in a scaling solution. In this case, near scale invariance is a generic feature of the theory if the field rolls in a steep self interaction potential.

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I. INTRODUCTION

The inflationary scenario is currently the favored model for the evolution of the very early universe [1, 2, 3, 4, 5, 6]. Inflation arises whenever the universe undergoes a phase of accelerated expansion and was originally introduced to solve a number of perceived problems with the hot big bang model of the universe, including the flatness, horizon and monopole problems [5]. More importantly, however, inflation is currently the favored model for large-scale structure formation since it can create a scale invariant spectrum of primordial density fluctuations, which provide the seeds of cosmic structure [7]. In the simplest versions of the scenario, inflation is realized by a scalar field, the inflaton, whose kinetic energy is negligible when compared to its potential energy such that $\dot{\phi}^2 < V$ [6]. This is typically called, slow-roll inflation.

Given the importance of inflation and that it occurred in the early stages of the universe's evolution, in potentially high curvature and density regimes, it is natural to investigate connections between inflation and quantum cosmology. This has recently been done in the context of Loop Quantum Cosmology (LQC), which is the application of Loop Quantum Gravity (LQG) to symmetric states (for reviews, see Refs. [8, 9, 10]). In particular LQC gives rise to a "semiclassical" regime in which the standard equations become modified by non-perturbative quantum geometrical effects. These semiclassical equations have been employed to study potential connections between LQC and inflation, and a number of important results have been obtained. For example in the context of a universe sourced by a minimally coupled scalar field, the semiclassical modifications cause an anti-frictional effect which accelerates the field along its self-interaction potential. In principle this effect can push the field up its potential and set the initial conditions for subsequent slow-roll inflation [11, 12, 13, 14, 15, 16].

The most striking feature of LQC, however, is that the anti-frictional effect also causes the universe to undergo an inflationary period [18]. In contrast to standard slow-roll inflation, where inflation is driven by the self-interaction potential of the scalar field, in LQC the inflationary phase is now driven by quantum geometrical effects. Moreover, it is possible to show that this period of inflation will occur independently of any particular form of the potential [17]. LQC therefore naturally predicts that the universe must evolve through an inflationary era, irrespective of whether this era is followed by a phase of standard slow-roll inflation or not.

It is natural to ask therefore, whether or not this period of LQC inflation is able either to replace or to supplement standard inflation, and what its observational signatures would be. In order to answer this question one must consider both the number of e-folds of inflation which the LQC phase can give rise to, and the spectrum of perturbations which this phase will produce.

The first of these issues has been addressed previously [19] and the conclusion was found to depend both on initial conditions and the value of a particular quantization ambiguity parameter labeled j [20]. In order to solve the problems of the hot big bang model, the required value of the parameter j is very large. That LQC inflation can replace standard inflation therefore seems disfavored, given that smaller values of the parameter j can be argued to be more natural than larger ones [11].

The issue concerning the spectrum of perturbations produced by the LQC inflationary phase, which might leave a signature of this phase, is a more subtle one. An important point is that during the LQC inflationary phase, not only does the growth of the scale factor accelerate, but the Hubble parameter also grows, $\dot{H} > 0$. Hence this phase is actually a superinflationary one, and experience from standard inflation suggests that we should expect the spectrum of perturbations to be strongly blue tilted ($n_s > 1$, where n_s is the spectral index). A recent study finds, however, that the LQC inflationary scenario can produce a nearly scale invariant

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spectrum of perturbations [21]. The study also finds that a generic prediction is that the spectrum will be slightly blue tilted, in contrast to most slow-roll models which have a small red tilt, and that this result is robust, being independent of ambiguities in the quantization scheme. This might lead one to believe that near scale invariance with a small blue tilt is a generic and observationally falsifiable result of LQC, in contrast with standard inflation where there is a large amount of freedom in the value of the spectral index associated with the form of the potential. The calculation of the power spectrum in Ref. [21], however, uses the so called direct method [22]. This method is not the standard one which is normally invoked for calculating the power spectrum of slow-roll inflation, but it is argued in Ref. [21] that the use of the direct method is more natural within LQC because of the minimum natural length scale introduced by LQC, and the lack of a general expression for the stress energy tensor in LQC.

Two further important aspects of the calculation in Ref. [21] are important to note. First, it assumes that the effective equation of state, $w = p/\rho$, for the universe as a whole is given by $w \approx -1$. This can only be true either at the end of the superinflationary phase, or under severe fine tuning of the model's parameters. Secondly, it assumes that the background spacetime in which the scalar field lives is unperturbed, and considers only perturbations in the scalar field. At the present time this is a necessary assumption, as the modified semi-classical equations of LQC are only known for an unperturbed background. This assumption is technically invalid as it clearly violates Einstein's field equations, however, whether it proves to be a useful approximation remains to be seen. Experience from other applications of perturbation theory in the early universe suggest that in some cases this approximation is very useful. For example in standard slow-roll inflation a calculation of the spectrum of scalar field perturbations, using this approximation, can be applied to produce an accurate estimate for the spectrum of the comoving curvature perturbations, which is ultimately the important quantity for observations [7, 23]. In other situations it is less useful, for example in the ekpyrotic scenario the application of a similar procedure to that used for slow-roll inflation yields erroneous predictions (see [24, 25]).

In this short note, we readdress the question of whether a scale invariant spectrum of primordial perturbations can be produced during the LQC superinflationary era using methods commonly employed in inflationary cosmology. Moreover, we do not make any assumptions about the universe's expansion rate, rather allowing it to be determined by the LQC dynamics. We continue to use the approximation in which the background spacetime is unperturbed and therefore focus on the spectrum of scalar field perturbations produced in this approximation.

We proceed as follows. In section II we introduce the semiclassical regime, we summarize previous results which we require in section III, and our new results are presented in sections IV and V. Finally we conclude in section VI with a summary of what we have learned, and a discussion of how the results may be useful in the future.

II. SEMI-CLASSICAL DYNAMICS

LQC is based on a Hamiltonian formulation of General Relativity. The dynamics is therefore governed by a Hamiltonian constraint equation which we can present schematically as $\mathcal{H}_{\rm gravity} + \mathcal{H}_{\rm matter} = 0$, where we have indicated that the constraint consists of a gravitational and a matter part.

The effective or semiclassical equations of LQC arise by incorporating into the classical Hamiltonian non-perturbative quantum effects from the underling LQG quantization procedure. A number of approaches have been taken to derive and verify the resulting effective Hamiltonian [26, 27, 28, 29]. A robust result of these approaches is the introduction of modifications which come from the quantization of inverse metrical quantities. In general such corrections occur both in the matter and the gravitational parts of the constraint, but most attention (with the exception of [28]) has so far been focused on corrections to the matter Hamiltonian which give rise to the superinflationary effect. Let us now consider this Hamiltonian in more detail when the matter source is a scalar field.

A crucial first step in formulating LQG is to rewrite canonical gravity in terms of Ashtekar variables, which are the densitised triad E^a_i and the Ashtekar connection A^i_a where $E^a_i = e^a_i/|\det e^b_i|$, with $e^a_i e^b_i = q^{ab}$ with q_{ab} the spatial metric, and $A^i_a = \Gamma^i_a + K^i_a$ with Γ the spin connection and K the extrinsic curvature. The indices run from one to three. When written in Ashtekar variables the matter Hamiltonian for a general spacetime becomes

$$\mathcal{H}_\phi = \frac{p_\phi^2}{2\sqrt{|\det E^c_j|}} + \frac{E^a_i E^b_i \partial_a\phi\partial_b\phi}{2\sqrt{|\det E^c_j|}} + \sqrt{|\det E^c_j|}\, V(\phi) \ . \tag{1}$$

The terms in this Hamiltonian which involve inverse expressions cannot be quantized in a straight forward manner and must first be regularized by a procedure introduced by Thiemann [30, 31]. The expressions which result from this procedure are rather complicated, and in particular are subject to an number of quantization ambiguity parameters.

Here we are interested in isotopic LQC and in this case the matter Hamiltonian reduces to

$$\mathcal{H}_\phi = \frac{1}{2}\frac{p_\phi^2}{\sqrt{|\det E^c_j|}} + \sqrt{|\det E^c_j|}\, V(\phi)\,, \tag{2}$$

where we have set the gradient terms, which would violate isotropy, to zero. We are also assuming, for simplicity, that E represents the isotopic triad. In this setting the only inverse term is the inverse volume ($|\det E^c_j|^{-1/2}$), which classically is simply a^{-3} in terms of metric variables. Quantum mechanically however, this term must be quantized following Thiemann's prescription (for details see [32]).

The spectrum for the inverse volume can be calculated exactly in isotropic LQC, but because of the regularization the answer depends on a number of ambiguity parameters [20]. Above the scale of discreteness set by $a_i = \sqrt{\gamma}\ell_{\rm pl}$, where $\gamma = 0.27$ is the Barbero Immizi parameter, spacetime can be considered to be continuous, and the inverse spectrum can be approximated by a continuous function. There is a second scale of importance, however, which is set by a second quantity, $a_* = a_i\sqrt{j/3}$, where j, which takes half integer values, is one of the quantization ambiguities. Above this scale the eigenvalues of the inverse operator follow the classical values, while below it they are radically different. In fact, where the classical inverse volume is infinite, i.e. at the classical singularity, in LQC the inverse volume is zero. We find therefore that if $a_* > a_i$, which implies $j > 3$, then there is an overlap between the regions in which the inverse volume can be approximated by a continuous function, and where the inverse volume deviates significantly from the classical expression. This is the semiclassical regime, and the semiclassical matter Hamiltonian is simply arrived at by replacing the inverse volume term in Eq. (2) by the continuous approximation function.

The function which approximates the spectrum of the inverse volume is given by $a^{-3}D(q)$, where $q \equiv (a/a_*)^2$ and

$$\begin{aligned} D(q) \;=\; & \Big\{ \frac{3}{2l} q^{1-l} \left[(l+2)^{-1} \left((q+1)^{l+2} - |q-1|^{l+2} \right) \right. \\ & - \frac{1}{1+l} q \left((q+1)^{l+1} \right. \\ & \left.\left. - \mathrm{sgn}(q-1)|q-1|^{l+1} \right) \right] \Big\}^{3/(2-2l)} , \end{aligned} \tag{3}$$

where l is another quantization ambiguity. From considerations of the regularization procedure within LQC, l is constrained to the range $0 < l < 1$, while from considerations of the procedure within the full theory of LQG it must take a discrete set of values given by $l_k = 1 - (2k)^{-1} \geq 1/2$, where k is an integer [20]. The expression for $D(q)$ for $a \ll a_*$ can be approximated by $D(q) \approx (3/(1+l))^{3/(2-2l)} q^{3(2-l)/2(1-l)}$. It has a global maximum at $a \approx a_*$ and falls monotonically to $D = 1$ for $a > a_*$. Hence the classical inverse volume, a^{-3} is recovered for $a \gg a_*$.

Replacing the inverse volume with this function one arrives at the semiclassical Hamiltonian, which we give here in terms of the scale factor including both the matter and gravitational parts:

$$\mathcal{H} = -\frac{3}{8\pi\ell_{\rm pl}^2}\,\dot{a}^2 a + \frac{1}{2} D a^{-3} p_\phi^2 + a^3 V = 0 \ . \tag{4}$$

We can now derive the semiclassical equations of motion.

Using this Hamiltonian density and considering $\dot\phi = \partial\mathcal{H}/\partial p_\phi$ we find that $p_\phi = a^3\dot\phi/D$, and hence $\mathcal{H}_\phi + a^3\dot\phi^2/2D + a^3 V$. Then on dividing the Hamiltonian constraint by a^3, we arrive at the modified Friedmann equation

$$H^2 \equiv \left(\frac{\dot a}{a}\right)^2 = \frac{8\pi\ell_{\rm pl}^2}{3}\left[\frac{1}{2} D^{-1}\,\dot\phi^2 + V(\phi)\right] \ . \tag{5}$$

The other dynamical equations can also be found using, $\dot p_\phi = -\partial\mathcal{H}/\partial\phi$,

$$\ddot\phi + 3H\left(1 - \frac{1}{3}\frac{d\ln D}{d\ln a}\right)\dot\phi + D\,\frac{dV(\phi)}{d\phi} = 0 \ , \tag{6}$$

and combining Eq. (6) with (5) we also find

$$\dot H = -4\pi\ell_{\rm pl}^2 \frac{\dot\phi^2}{D}\left(1 - \frac{1}{6}\frac{d\ln D}{d\ln a}\right) \ . \tag{7}$$

III. BACKGROUND EVOLUTION

Before moving on to considering perturbations of the scalar field we require an understanding of the background dynamics. In particular we confine ourselves to the regime $a \ll a_*$, where analytic progress can be made.

In this regime, we write the correction function as $D = D_* a^n$ with $n = 3(2-l)/(1-l)$ (hence $6 < n < \infty$) and $D_* = (3/(1+l))^{3/(2-2l)} a_*^{3(l-2)/(1-l)}$. From Eq. (7) it can be seen that the universe undergoes superinflationary expansion, $\dot H > 0$, for $n > 6$, independently of the form of the self-interaction potential. We will be interested in the cases where the ratio $\sqrt{2D}\,H/\dot\phi = $ constant. This comprises the cases of a massless scalar field and of a scaling solution (when the ratio of kinetic to potential energy is a constant). In both these cases, the evolution can be solved exactly, and the scale factor undergoes power law growth. When we come to deal with the perturbed equations, we will find it more convenient to work with conformal time $dt = a\,d\tau$, and so we give the background evolution using this time variable.

A. Massless scalar field

Considering a massless scalar field, Eqs. (5), (6) and (7) can be solved to yield $a^{2-n/2} = a_{\rm init}^{2-n/2} + A^{2-n/2}(\tau_{\rm init} - \tau)$ where

$$A = \left[\left(\frac{n}{2} - 2\right)\left(\frac{4\pi\ell_{\rm pl}^2}{3} D_* \phi_{\rm init}^{\prime 2} \frac{a_{\rm init}^4}{D_{\rm init}^2}\right)^{1/2}\right]^{2/(4-n)} , \tag{8}$$

which is positive-definite for $n > 4$. It is convenient to rescale time to absorb the constant term into our definition of conformal time such that

$$a = A\,(-\tau)^p \,, \tag{9}$$

where $p = 2/(4-n) < 0$. It is worth pointing out that τ is negative and increasing for an expanding universe and decreasing for a contracting universe.

B. Scaling Solution

A second way to achieve power law growth in the regime $a \ll a_*$ is for the field to roll in a self interactive potential of the form [33]

$$V(\phi) = V_0\,|\phi|^\beta\,, \tag{10}$$

with $\beta > 0$.

The analogous scaling solution in the classical regime, which is exponential in form, is very important in understanding the production and evolution of perturbations in the standard single field inflationary scenario, and we expect that this scaling solution has the same degree of importance in the LQC scenario.

Using a rescaled conformal time, the growth of the scale factor and the field are then determined by the expressions

$$a = A\,(-\tau)^p\,, \qquad \phi = F\,(-\tau)^v\,, \tag{11}$$

where $v = np/2$, $p = -4/(n\beta+4) < 0$. V_0 is related to the constants A and F and the powers v, n and p, however, this relation is not important in what follows. Obviously, the constant A, does not need to take the same value as in Eq. (8).

IV. PERTURBATION THEORY

If we are to fully understand the evolution of cosmological perturbations in LQC, we must perturb both the gravitational and the matter sectors of the theory, about the homogeneous background. So far, however, the quantization procedure in LQC has only been performed for homogeneous spacetimes, and not for the perturbed cases. This means that the full perturbed semiclassical equations have so far not been derived. In their absence, we may adopt a more modest approach. This is, to assume that the background spacetime is unperturbed, but to allow perturbations in the scalar field. The assumption is valid for cases in which perturbations of spacetime are much smaller than those of the matter source, or equivalently where the matter perturbations have a negligible effect on the background spacetime. In this case we can calculate the power spectrum of the scalar field's perturbations on super-horizon scales produced from quantum mechanical fluctuations.

Ultimately the quantity which is relevant to observations after the inflationary era is the power spectrum of the comoving curvature perturbation. We comment how this might be calculated from the spectrum for the scalar field perturbations in the discussion section.

To follow even the modest approach and allow inhomogeneities in the scalar field, we must include a gradient term in the matter Hamiltonian, $\mathcal{H}_\phi$. This term, strictly speaking, violates homogeneity but we will assume that the effect on the background spacetime is sufficiently small that it can be neglected. Including this extra term, the matter part of Eq. (4) becomes

$$\begin{aligned}\mathcal{H}_{\phi+\delta\phi} \;=\; & \frac{1}{2}D(a)a^{-3}p^2_{\phi+\delta\phi} \\ & +\frac{1}{2}\,a\,G(a)\,\delta^{ij}\partial_i(\phi+\delta\phi)\partial_j(\phi+\delta\phi) \\ & +a^3V(\phi+\delta\phi)\,.\end{aligned} \tag{12}$$

In this expression we have introduced a correction function $G(a)$, which we expect to arise due to the term $E^a_iE^b_i\partial_a\phi\partial_b\phi/|\det E^c_j|^{-1/2}$ in Eq. (1) which involves inverse quantities, and must be regularized in a similar manner to the inverse volume term. This term has so far not been calculated within LQC, and to attempt to do so here is beyond the scope of this note. Indeed in the previous study of perturbations in LQC this term was assumed to be unity. The relevant regularization procedure for this term is closely connected to that for the inverse volume, and we anticipate it to have a similar form, in particular we assume that $G = 1$ in the classical regime, but has a region for small values of the scale factor where $G \propto a^r$, where r will depend on a new quantization parameter. In our study we will again assume, as in previous studies, that $r = 0$. However, our method can easily be generalized to take account of a non zero r and in the section VI we discuss how this would affect our results.

Using Eq. (12), the unperturbed Eqs. (5)–(7) are unaltered, but we have the additional perturbation equation

$$\delta\phi'' = \left[-2\frac{a'}{a} + \frac{D'}{D}\right]\delta\phi' + D\,\left[\nabla^2 - a^2\frac{d^2V}{d\phi^2}\right]\delta\phi\,, \tag{13}$$

where a prime means differentiation with respect to conformal time τ. Using conformal time is helpful because it allows us to write Eq. (13) in the particularly simple form

$$u'' + \left(Dk^2 + m^2_{\rm eff}\right)u = 0\,\,, \tag{14}$$

where u is defined as $u = aD^{-1/2}\delta\phi$, and

$$m^2_{\rm eff} = -\frac{(aD^{-1/2})''}{aD^{-1/2}} + a^2D\frac{\partial^2V}{\partial\phi^2}\,, \tag{15}$$

is the effective mass of the field u.

In an equivalent approach, the scalar field equation in the semiclassical LQC regime in the absence of metric perturbations, can be derived from an effective action. In terms of conformal time, the action can be written as

$$S = \int d\tau\, d^3\mathbf{x}\mathcal{L} = \int d\tau\, d^3\mathbf{x}a^4\left(\frac{1}{2}\,\frac{\phi'^2}{D\,a^2} - V\right)\,. \tag{16}$$

Adding the gradient term $-\delta^{ij}\partial_i\phi\partial_j\phi/2a$ to the quantity within brackets and including a linear perturbation in the field around its background solution we find that the perturbed part of S can be written as

$$\delta S = \frac{1}{2}\int d\tau\, d^3\mathbf{x}\left(u'^2 - D\,\delta^{ij}\partial_i u\,\partial_j u - m_{\rm eff}^2 u^2\right)\ , \quad (17)$$

and when varied, this action also leads to Eq. (14).

V. POWER SPECTRUM

The action for u (17) is now formally equivalent to that of a scalar field with a variable mass term, and a D term multiplying the gradient part. In order to calculate the spectrum of the perturbations, produced during the super-inflation due to quantum fluctuations, we must consider the field theory associated with the field u.

The momentum canonically conjugate to u is given by

$$\pi(\tau, x) = \frac{\partial\mathcal{L}}{\partial u'} = u' \quad (18)$$

The theory is then quantized by promoting u and π to operators which satisfy the usual commutation relations. We Fourier decompose $\hat{u}$ to give

$$\hat{u} = \int \frac{d^3k}{(2\pi)^{3/2}}\left[w_k(\tau)\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + w_k^*(\tau)\hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}\right]\ , \quad (19)$$

where w_k are mode functions which satisfy the same equation as u

$$\frac{d^2 w_k}{d\tau^2} + \left(Dk^2 + m_{\rm eff}^2\right)w_k = 0\ . \quad (20)$$

In order to have a well defined field theory, we must also ensure that w_k is such that the creation and annihilation operators, $\hat{a}_{\mathbf{k}}^\dagger$ and $\hat{a}_{\mathbf{k}}$, satisfy the usual commutation relations for bosons. This means that w_k must satisfy the Wronskian condition

$$w_k^*\frac{dw_k}{d\tau} - w_k\frac{dw_k^*}{d\tau} = -i\,. \quad (21)$$

In general, however, this condition does not give rise to a unique choice for w_k, instead it allows a set of possible choices corresponding to a set of different Fock representations. In the cosmological context a unique choice is normally determined by considering a limit in which the time dependence of the scale factor can be neglected, and hence where the physics ought to reduce to that of Minkowski space. In this limit w_k is normalized to select only the advanced solution. Once the initial condition is selected and the Wronskian condition met, a vacuum state is defined which is annihilated by all $\hat{a}_{\mathbf{k}}$, such that $\hat{a}_{\mathbf{k}}|0\rangle = 0$.

The power spectrum of fluctuations about this vacuum state is defined by the vacuum expectation value such that

$$\langle u_{\mathbf{k}}u_{\mathbf{l}}^*\rangle = \frac{2\pi^2}{k^3}P_u\delta^{(3)}(\mathbf{k}-\mathbf{l})\ , \quad (22)$$

where we have implicitly Fourier decomposed the field perturbation $\delta\phi$, and defined $u_{\mathbf{k}} = a\delta\phi_{\mathbf{k}}$. Using Eq. (19) we find

$$\langle u_{\mathbf{k}}u_{\mathbf{l}}^*\rangle = |w_k|^2\delta^{(3)}(\mathbf{k}-\mathbf{l})\ , \quad (23)$$

and hence that the power spectrum is given by

$$\mathcal{P}_u = \frac{k^3}{2\pi^2}|w_k|^2\ . \quad (24)$$

We now proceed to derive the form of the power spectra for the two cases under study.

A. Massless field

As we have seen, during the semiclassical phase we have $D = D_* a^n$, and we have power law growth with $a = A(-\tau)^p$ where $p = 2/(4-n)$. Inserting this into Eq. (20) we obtain

$$\frac{d^2 w_k}{d\tau^2} + \left(D_*\,A^n(-\tau)^{np}k^2 + \frac{m_{\rm eff}^2\tau^2}{\tau^2}\right)w_k = 0\,. \quad (25)$$

For the massless case, using Eq. (15), we find

$$m_{\rm eff}^2\,\tau^2 = -p(p-1)\,. \quad (26)$$

The general solution admitted by Eq. (25) is,

$$w_k(\tau) = c_1\sqrt{-\tau}\,J_{|\nu|}(x) + c_2\sqrt{-\tau}\,Y_{|\nu|}(x)\,, \quad (27)$$

where $J_{|\nu|}(x)$ and $Y_{|\nu|}(x)$ are Bessel functions of the first and second kind respectively and we have defined

$$\nu = -\frac{\sqrt{1-4\,m_{\rm eff}^2\,\tau^2}}{2+np}\,, \quad (28)$$

and

$$x = \alpha\,k\,(-\tau)^{(2+np)/2} = \left|\frac{2p}{2+np}\right|\frac{\sqrt{D}\,k}{aH}\,. \quad (29)$$

with $\alpha = 2\sqrt{D_*A^n}/|2+np|$ and $x>0$. We normalize this solution such that the Wronskian condition (21) is satisfied which in general gives

$$w_k(\tau) = \sqrt{\frac{\pi}{2|2+np|}}\left(d_1\sqrt{-\tau}\,H_{|\nu|}^{(1)}(x) + d_2\sqrt{-\tau}\,H_{|\nu|}^{(2)}(x)\right)\,, \quad (30)$$

where d_1 and d_2 are constants subject to the condition $|d_1|^2 - |d_2|^2 = 1$ and $H_{|\nu|}^{(1)}(x)$ and $H_{|\nu|}^{(2)}(x)$ are Hankel functions of the first and second kind, respectively. Moreover, the Hankel and Bessel functions are related through the expressions: $H_{|\nu|}^{(1)}(x) = J_{|\nu|}(x) + i\,Y_{|\nu|}(x)$ and $H_{|\nu|}^{(2)}(x) = J_{|\nu|}(x) - i\,Y_{|\nu|}(x)$. We now consider the small wavelength limit in which the wavelength of the mode functions is far inside the cosmological horizon, and where we might expect a Minkowski form for the mode functions. This limit corresponds to, $x \gg 1$, and the asymptomatic form of Eq. (30) is

$$w_k(\tau) = \frac{(-\tau)^{-np/4}}{\sqrt{|2+np|\,\alpha k}} \Big(d_1 \, \exp(i\alpha k(-\tau)^{(2+np)/2}) + d_2 \, \exp(-i\alpha k(-\tau)^{(2+np)/2}) \Big) \, . \qquad (31)$$

In the standard inflationary scenario the analogous solution reduces to two plane waves propagating in opposite directions in time, only the advanced solution is selected and $w_k(\tau) = e^{-ik\tau}/\sqrt{2k}$ in this limit. In our case the solution only has the same form as flat spacetime when $n = 0$, i.e. when the universe is classical. The two components to our solution however still represent advanced and retarded solutions, and by analogy we select only the advanced solution, this means we set $d_1 = 1$ and $d_2 = 0$. This can be justified by considering a mode whose wavelength remains well inside the cosmological horizon throughout the superinflationary evolution. Our normalization is then consistent with the Minkowski limit once super-inflation has ended. Moreover we note that ultimately our interest is in the k dependence of the solution in the large wavelength limit, which is unaltered by the normalization as long as the Wronskian condition is satisfied. That the solution does not reduce to the Minkowski limit however, already suggests that there are going to be clear differences in the evolution of perturbations with respect to the standard case whenever a geometric correction to the kinetic term of the field occurs.

We can now look at the long wavelength limit of our properly normalized mode functions. The long wavelength limit is given by $k \ll 1$, and for a specific finite time τ this corresponds to $x \ll 1$, and hence to wavelengths well outside the effective horizon. In this limit we have

$$J_{|\nu|}(x) \rightarrow \frac{1}{\Gamma(|\nu|+1)} \left(\frac{x}{2}\right)^{|\nu|} , \qquad (32)$$

$$Y_{|\nu|}(x) \rightarrow -\frac{\Gamma(|\nu|)}{\pi} \left(\frac{x}{2}\right)^{-|\nu|} . \qquad (33)$$

At this point a few comments are in order. As we have seen before, in the massless field case under study the growth power p is negative, which means that the quantity $x \propto (-\tau)^{(2+np)/2} \propto (-\tau)^{2p}$ is an increasing function as $\tau \rightarrow 0$. Therefore, though the $Y_{|\nu|}(x)$ solution is the dominant one at early times, it is decreasing in nature and soon becomes sub dominant with respect to the increasing $J_{|\nu|}(x)$ solution. A related consequence of the growth power of x, $2p$, being negative is that as opposed to the standard inflationary scenario where the modes exit the effective horizon $1/aH$ during inflation, here, during super inflation driven by quantum effects, the modes *enter* the effective horizon given by $\sqrt{D}/aH$. Conversely, in the situation in which the universe is undergoing a collapsing evolution, modes eventually *exit* the horizon. We will see that this not necessarily the case for the scaling solution.

While this is interesting, it raises a serious interpretational issue. In standard inflation, the short wavelength limit is the same as the $\tau \rightarrow -\infty$ limit, so all wavelengths can be considered to be small compared with the cosmological horizon at the earliest times. In this limit it is natural to assume that the small scale perturbations are governed by quantum mechanics and the normalization is performed in this limit. As the expansion proceeds however the physical wavelength of the modes is increased, or equivalently the cosmological horizon size is decreased. The modes are pushed outside the horizon as this behavior proceeds. The modes effectively become classical, and the spectrum calculated in this limit can also be interpreted as a classical spectrum. In the case at hand, however, this is no longer true, and it is not clear whether we can interpret the spectrum calculated on long wavelengths as a classical one.

Taking this caveat on board, let us nevertheless proceed with the calculation. Using only the dominant part of Eq. (30) with the help of Eq. (33), the power spectra Eq. (24) reduces to

$$\mathcal{P}_u = \frac{1}{4\pi} \left| \frac{p}{2+np} \right|^{1-2|\nu|} \left(\frac{\Gamma(|\nu|)}{\pi} \right)^2 \times \frac{a^2 H^2}{D^{3/2}} \left(\frac{\sqrt{D}\,k}{aH} \right)^{3-2|\nu|} \propto k^{3-2|\nu|} \, (-\tau)^{1-|\nu|(np+2)} \, , \qquad (34)$$

which for our massless field example (where $\nu = -n/8$ from Eq. (28)), yields

$$\mathcal{P}_u \propto k^{3-2|\nu|} \, (-\tau)^{2(n-2)/(n-4)} \, . \qquad (35)$$

Then using $\mathcal{P}_\phi = D\mathcal{P}_u/a^2$, we find

$$\mathcal{P}_\phi \propto \frac{H^2}{D^{1/2}} \left(\frac{\sqrt{D}\,k}{aH} \right)^{3-2|\nu|} \propto k^{3-2|\nu|} \, (-\tau)^{1+p(n-2)-|\nu|(np+2)} \, , \qquad (36)$$

which for the massless case, turns out to be time independent which tell us that the evolution of the scalar field perturbation is frozen on super horizon scales. We can also conclude that, for this case, one obtains scale invariance of the scalar field perturbation only when $n = 12$.

B. Scaling solution

We can follow the same procedure for the self interaction potential with a scaling solution. In this case, using Eq. (15) we obtain

$$m_{\rm eff}^2 \tau^2 = -2 + (3-2n)p + \frac{1}{2}(6+2n-n^2)p^2 \, , \qquad (37)$$

and for the quantity ν we have from Eq. (28)

$$\nu = -\frac{\sqrt{9 - 12p + 8np - 12p^2 - 4p^2 n + 2n^2 p^2}}{2 + np}\,, \quad (38)$$

where $p = -4/(n\beta+4)$. In this case we do not necessarily encounter the same behavior found in the massless case where the modes enter the horizon during the super inflationary phase. In fact, for $\beta > 2 - 4/n$ we have that x is now decreasing as $\tau \to 0$. Hence, for these values of β, the dominant solution is always the $Y_{|\nu|}(x)$ function and the modes exit the effective horizon during the evolution.

In the limit of large β we have that p approaches zero and hence we have $\nu \to -3/2$ which gives scale invariance. It is interesting to note that, in the limit of large β, the power spectrum is scale invariant regardless of n (or the quantization parameter l). This is easy to understand because in this limit Eq. (20) approaches

$$w_k'' + \tilde{k}^2 w_k - \frac{2}{\tau^2}\, w_k = 0\,, \quad (39)$$

with $\tilde{k}^2 = D_* A^n k^2$, which is of similar form to the analogous equation in the case of slow-roll inflation. Because the equation takes this form, the $x \ll 1$ limit is identical to that for standard inflation and the Minkowski space limit is recovered, removing another conceptual problem. The multiplicative factor in $\tilde{k}^2$ affects the normalization of the power spectrum but not the scale dependence, which is independent of n. It is also interesting that no fine tuning of the n (or l) parameter is required and that the solution is stable in the sense that $\beta \gg 1$ corresponds to background solutions which are stable to linear homogeneous perturbations [33]. This limit corresponds to the condition of a very steep potential which means that a, and hence D, are nearly constant despite H varying. From Eq. (15) we see that this means in this case the potential term is the most significant part of $m_{\rm eff}^2$.

Using Eqs. (34) and (36) we see that in the limit of large β we have for the power spectrum,

$$P_\phi = \left(D_* A^{n+4}\right)^{-1/2} (2\pi\tau)^{-2}\,, \quad (40)$$

independent of n.

We conclude that nearly scale invariance is a natural prediction of the LQC universe sourced by a steep potential of the form given by Eq. (10). As will be discussed below, we must not over emphasize this result as it may change if metric perturbations are significant.

There is, however, an additional solution for a particular fine tuned value of β which gives $\nu = 3/2$ and hence also scale invariance. Indeed, as we decrease the value of β, we see that at $np = -2$ or $\beta = (2n-4)/n$, the value of ν blows up to $-\infty$ and switches sign. As β approaches zero, p approaches -1 and consequently $\nu \to \sqrt{9 - 12n + 2n^2}/(n-2)$ which is always between $\sqrt{2} < \nu < 3/2$, therefore, ν must cross the value $\nu = 3/2$ at small β. A scale invariant power spectrum is therefore possible for small β but subject to a severe fine tuning.

VI. DISCUSSION

In this article we have computed the power spectrum of the scalar field perturbations for two distinct situations. First we have considered the dynamics of a massless scalar field. We found the interesting behavior that the modes of the scalar field perturbations enter the effective horizon during the superinflationary phase in clear contrast with the evolution in standard slow-roll inflation. Scale invariance is possible in this case but at the cost of the fine tuning of the quantization parameter l. However we note that the required value $n = 12$ ($l = 2/3$) is not one of the values which are favored by consideration of the full theory.

An interesting question is whether allowing the parameter function G to vary from unity would modify the phenomenology. It is easy to see that the effect of the function G on the k dependence, is to change Eq. (28) such that $\nu = -(1 - 4m_{\rm eff}^2\tau^2)^{1/2}/(2 + np + rp)$, while $m_{\rm eff}$ remains unaltered. Hence there is an extra degree of freedom in this case, and it would be interesting to investigate whether scale invariance can be achieved without moving away from the preferred values of quantization parameters using this freedom.

Before leaving the massless case it is also worth noting that even in the limit $l = 0$, which represents exponential expansion $\dot{H} = 0$, we do not find a scale variant spectrum and we still have the problem of modes entering the horizon. This is in stark contrast with earlier work [21] which, at least in part, motivated our investigation. Indeed within our calculation even if we had imposed by hand that the background evolution was exponential (i.e. fixing $p = -1$ but leaving n free), we would have obtained scale invariance provided that $n = 0$ or $n = 12/5$. Again, in contrast with the previous study.

The fine tuning displayed above is evaded in the second situation we investigated, that of the scaling solution. By including a self interaction potential, we gain a degree of freedom that can be used to set the region of parameter space that offers the desired features of the inflationary scenario i.e. modes exiting the horizon and near scale invariant power spectrum of perturbations. Moreover it is clear that allowing the G function to vary will not affect this limiting behavior since the $p \to 0$ limit ensures G will be close to a constant.

At this point in the discussion it is useful to say something about when we might expect our calculation to be accurate, and how it could be applied to derive the comoving curvature perturbation. This quantity is useful since under very general circumstances it is conserved on super-horizon scales. Hence its spectrum calculated as modes leave the horizon during inflation is equal to the spectrum as these modes re-enter at a later time, when they account for the formation of cosmic structure. In our calculation we have simply calculated the scalar field spectrum during super-inflation, however, we would like to convert this result into the comoving curvature spectrum. To make this conversion we must pick a gauge in which we assume that the metric perturbations in the scalar field equation are sub dominant to the scalar field perturbations, as in this gauge our calculation will be accurate. If this is the spatially flat gauge then the conversion to the curvature perturbation would simply be given by $P_{\mathcal{R}} = (\mathcal{H}/\phi')^2 P_\phi$. To convert the spectrum in this way is the procedure used in Ref. [7] for standard inflation. However, until the full equations of gravitational and matter perturbations are known we will not know in what gauges (if any) the metric perturbations can be ignored, and hence how to convert our spectrum to the spectrum of curvature perturbations.

In addition to the inclusion of the background perturbations into our calculation, a further way to improve its accuracy consists of relaxing the assumption that $a \ll a_*$, and hence by using the full form of the function $D(a)$, rather than its asymptotic approximation. This would require the mode functions to be solved numerically, and it would be interesting to compare this approach with the analytic results derived here.

Nonetheless, our aim was not to establish a robust prediction for the spectral index from LQC inflation, but rather to illustrate three important points. First, subject to the approximations we have to make, a scale invariant spectrum is possible for LQC inflation even when the equation of state differs from $w \approx -1$. This is a great surprise considering our experience from standard inflation. Second, even when the universe is super-inflating the Fourier space modes of the scalar field perturbation are not necessarily pushed outside a suitably defined horizon. We encountered this type of behavior in the massless case. This again is unexpected considering standard inflation. Third, just like in slow-roll inflation, when calculated with the standard techniques, there is considerable freedom in the value of the spectral index from LQC inflation. This freedom is related to the form of the potential and, uniquely to LQC, it is also dependent on the choice of quantization ambiguities. In particular we expect that, if we consider potentials other than those that generate a scaling solution, the spectral index may be greater or less than unity.

We argue, therefore, that our calculation is an important towards understanding the phenomenology of LQC. In particular it highlights these three features which are likely to carry over to a full analysis including metric perturbations, and which deserve considerable attention. The calculation is also important given the likely complexity of the full perturbed equations. Once the full equations are known we will be able to determine when our calculation ought to provide an accurate answer, and in these cases it will provide a useful check on any spectrum of perturbations calculated using the full equations. As we have seen there are many subtleties in any calculation of a perturbation spectrum. It may even be that the approach of using the full equations to determine when background perturbations can be ignored and performing the calculation given here, is the only case in which the spectrum of perturbations from inflation in LQC can be determined using the standard techniques.

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